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The determination of elastic constants by piezo-electric methods

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THE DETERMINATION OF ELASTIC CONSTANTS BY PIEZO-ELECTRIC METHODS

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INTRODUCTION

The velocity of propagation of wave motions in an elastic body depends on the elastic constants of the material composing the body. It is possible by mathematical methods to solve for the relationship between the velocity of the various types of wave motion and the elastic constants whether the substance be isotropic or aeolotropic. It is possible, therefore, knowing the elastic constants, to find the velocity of wave motions in any given direction in the homogeneous, aeolotropic substance; or, vice versa, knowing a sufficient number of independent velocities, it is possible to calculate the elastic constants.

The purpose of this investigation is to determine the six independent elastic constants of quartz by means of an accurate measurement of the frequencies of vibrations in quartz plates when standing waves are produced between the surfaces of the plates.

THEORETICAL

The Theory

The characteristic equation

By the methods of tensor analysis one may express briefly and concisely the relationships between the stresses and the strains in a homogeneous, aeolotropic material. The standard tensor notation will be used in the following work. According to the usual convention, the repetition of any subscripts in a product or other mathematical operation indicates a summation of the term in which the repetition occurs such that the subscript index runs from 1 to 3 for all subscripts repeated.

The coordinate axes are indicated by x_1, x_2, and x_3 instead of X, Y, and Z. The stress tensor is denoted by $\mathcal{O}_{ij}$ and the strain tensor by Θ_{ab}. The stress components, $\mathcal{O}_{11}$, $\mathcal{O}_{22}$, and $\mathcal{O}_{33}$ represent tensions in the directions of the corresponding axes. The components $\mathcal{O}_{12}$, $\mathcal{O}_{13}$, etc., represent the shearing stresses, the first subscript denoting the direction in which the force acts and the second subscript denoting the plane, perpendicular to the direction indicated by the subscript, on which the force acts. The actual displacements occurring in the medium are denoted by u_1, u_2, and u_3.

The relation between the stresses and the strains in an aeolotropic medium is given by a generalized Hooke's law:

$$\mathcal{O}_{ij} = c_{ijab}\, \Theta_{ab}, \qquad \ldots\ldots\ldots\ldots\ldots \quad 1.$$

where the c's are the elastic constants of the medium.

Because of the symmetry of the stress and strain tensors, the ij and the ab subscripts of the elastic constants can be reversed in order without affecting the value of the constants. The pair of subscripts ij can also be interchanged with the pair ab. This fact is connected with the existence of a strain energy function.

The force per unit volume experienced by the stressed medium is

$$F_j = \rho \ddot{u}_j = \frac{\partial \mathcal{O}_{ij}}{\partial x_i} \qquad \ldots\ldots\ldots\ldots\ldots \quad 2.$$

where ρ is the density of the medium.

On substitution of $\mathcal{O}_{ij}$ from equation 1 into equation 2, the result is

$$\rho \ddot{u}_j = c_{ijab} \frac{\partial \Theta_{ab}}{\partial x_i} \qquad \ldots\ldots\ldots\ldots\ldots \quad 3.$$

Upon substitution of the expression for the strain tensor in terms of the displacements in the medium,

$$\Theta_{ab} = \frac{1}{2}\left(\frac{\partial u_a}{\partial x_b} + \frac{\partial u_b}{\partial x_a}\right), \ldots\ldots\ldots\ldots\ldots \quad 4.$$

equation 3 becomes

$$\rho \ddot{u}_j = \frac{1}{2}\left(c_{ijab} \frac{\partial^2 u_a}{\partial x_i \partial x_b} + c_{ijab} \frac{\partial^2 u_b}{\partial x_i \partial x_a}\right) \qquad \ldots\ldots\ldots\ldots \quad 5.$$

The subscripts a and b, being indices of summation, can be interchanged in any given term without altering the meaning. Therefore, remembering that $c_{ijab} = c_{ijba}$, the expression reduces to the differential equation of motion of the medium,

$$\rho \ddot{u}_j = C_{ijab} \frac{\partial^2 u_a}{\partial x_i \partial x_b} \quad \text{.................................} \quad 5.$$

The characteristic value differential equation for periodic motion can be obtained from equation 5 by the substitution, $u_j = \psi_j e^{\sqrt{-1}\omega t}$. The result is

$$C_{ijab} \frac{\partial^2 \psi_a}{\partial x_i \partial x_b} + \rho \omega^2 \psi_j = 0 \quad \text{.........................} \quad 6.$$

This equation is true for any periodic motion in a homogeneous medium of any shape.

<u>Thickness vibrations in an infinite plate</u>

The problem at hand is to obtain a solution of equation 6 for an infinite plate which has no external forces acting on it. Let the x_2 and x_3 dimensions of the plate be infinite, and let the x_1 dimension between the parallel surfaces of the plate be d. The boundary conditions are merely that the stress components acting across the boundary are zero.

A solution of the form $\psi_j = A_j \cos \frac{n\pi x_1}{d}$ satisfies the boundary conditions since the stresses are linear functions of $\sin \frac{n\pi x_1}{d}$, which is equal to zero when x_1 is equal to 0 or d. A_j is proportional to the amplitude of vibration, and n is the order of the harmonic of the particular mode of vibration. The solution, ψ_j, represents standing waves between the surfaces of the plate, the phase planes of the waves being parallel to these surfaces. Substitution of $\psi_j = A_j \cos \frac{n\pi x_1}{d}$ into equation 6 gives as the only condition that ψ_j be a solution,

$$-A_a \frac{n^2\pi^2}{d^2} C_{1ja1} + \rho \omega^2 A_j = 0$$

The expression $\frac{\rho\omega^2 d^2}{n^2\pi^2}$ may be set equal to κ^2 for brevity. Since $\frac{\omega}{2\pi}$ is equal to a fundamental resonance frequency of the plate, denoted by γ, then

$$\kappa^2 = 4\rho d^2\left(\frac{\gamma}{n}\right)^2 \qquad \text{7.}$$

The condition that ψ_j is a solution of the characteristic value differential equation now takes the form

$$A_a c_{1ja1} - \kappa^2 A_j = 0 \qquad \text{8.}$$

On expansion, equation 8 represents a set of three linear equations in the three unknown A's. Since the equations are homogeneous, there will be solutions only if the determinant of the coefficients is equal to zero. Thus the secular equation is

$$\begin{vmatrix} c_{1111}-\kappa^2 & c_{1121} & c_{1131} \\ c_{1211} & c_{1221}-\kappa^2 & c_{1231} \\ c_{1311} & c_{1321} & c_{1331}-\kappa^2 \end{vmatrix} = 0 \qquad \text{9.}$$

This equation gives the desired relationship between the elastic constants and the resonance frequencies of the infinite plate. Since the expansion of the determinant gives an equation of the third order in κ^2, there are, in general, three values of κ^2 which satisfy the equation, and therefore there are three fundamental resonance frequencies or normal modes of vibration, defined by equation 7. 9?

Since the equations 8 are homogeneous, ratios between the A's but no numerical values result when values of κ^2 are substituted into the equations. Thus, the frequency of vibration is independent of the amplitude of the vibration. The ratios of the A's define the normal modes of vibration of the plate.

If the elastic constants are known, the frequency of the normal modes of vibration can be found through the solution of a cubic equation. However, if the elastic constants are to be determined, the resonant frequencies of several plates having various orientations with respect to the material must be known, since a single determinant of the form 9 does not, in general, involve all the elastic constants. At least as many independent modes of vibration must be known as the number of independent elastic constants that are to be evaluated. Since the elastic constants, in general, are a function of temperature, it is necessary, of course, that the plates all be at the same temperature when their frequencies are determined.

Transformation of elastic constants

The determinant 9 is true for any random orientation of the infinite plate with respect to the natural axes of symmetry of the substance of which the plate is composed. Since elastic constants of a crystalline substance are logically given with respect to the natural crystallographic axes, it is important to consider how the elastic constants are transformed as the axes are rotated.

The elastic constants, being tensors of the fourth rank, transform with a change of coordinates according to the equation

$$c_{abcd} = l_{a\alpha} l_{b\beta} l_{c\gamma} l_{d\delta} c'_{\alpha\beta\gamma\delta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \; 10.$$

where the l's are the direction cosines between the new and the old coordinate axes. Thus, the elastic constants for any orientation can be expressed as linear functions of the constants for the normal orientation.

The λ-secular equation

The theoretical development so far presented is due to Atanasoff and Wilson[1]. A somewhat similar secular equation was developed and used by Koga[9].

If one is interested only in the frequencies of vibration and not in the modes it is possible to modify the form of the secular equation 9 so that the terms form a tensor of the second rank instead of one of the fourth rank. This form of the secular equation is due to Christoffel and has been quoted and used by Love[2], Mason[3], etc.

Upon substitution of equation 10 into equation 8 there results

$$l_{1\alpha} l_{j\beta} l_{a\gamma} l_{1\delta}\, C'_{\alpha\beta\gamma\delta}\, A_a = \kappa^2 A_j$$

Multiplying the above equation by the direction cosine l_{jm} and carrying out the indicated summation over j gives

$$l_{1\alpha}\, \delta_{m\beta}\, l_{a\gamma} l_{\delta 1}\, C'_{\alpha\beta\gamma\delta}\, A_a = \kappa^2 l_{jm} A_j$$

Here $\delta_{m\beta}$ is Kronecker's δ, having the property that it is equal to 0 for m not equal to β and it is equal to 1 for m equal to β. Therefore, setting $B_m = l_{jm} A_j$ the system of equations becomes

$$l_{1\alpha} l_{1\delta}\, C'_{\alpha m\gamma\delta}\, B_\gamma - \kappa^2 B_m = 0$$

Evidently the secular equation has the form

$$\begin{vmatrix} \lambda_{11}-k^2 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22}-k^2 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33}-k^2 \end{vmatrix} = 0 \qquad \text{.............. 11.}$$

where

$$\lambda_{ij} = l_{1\alpha} l_{1\delta} c'_{\alpha ij\delta} \qquad \text{.............. 12.}$$

Since it is easier to obtain the λ-'s, for a given orientation of the plate, through equation 12, than it is to transform the elastic constants, the λ-secular equation is often the most convenient form for actual use, but it gives no direct information as to the mode of vibration.

Behavior of finite plates

A plate having two infinite dimensions is purely theoretical. Any experimental work on the determination of the elastic constants must, of course, make use of finite samples of the material. Because of its complexity, the problem of the normal modes of vibration of finite plates has never been solved rigorously. However, by the use of certain restrictions, the finite plate can be made to approximate closely the behavior of an infinite plate.

The introduction of finite lateral dimensions has several effects on the behavior of the plate. The first effect is to introduce other natural resonance frequencies with all their harmonics, produced by the setting up of standing waves between the lateral surfaces. Modes of vibration whose frequencies are close to the same value tend to couple together and to perturb the frequency of each other.

According to the principle of Saint-Venant, the redistribution of stresses on a small surface has a negligible effect on the stresses at distances large in comparison with the linear dimensions of the surface on which the forces are changed. It is evident also that if the lateral dimensions are large, the normal and lateral modes of vibration will differ greatly in frequency, and therefore the coupling will be less. Hence, the smaller the thickness dimension of the plate in comparison with the other dimensions, the closer the behavior of the finite plate will approach the behavior of an infinite plate.

An infinite plate may be approximated by making the lateral dimensions very large, but there are obvious limitations in this direction. The use of very thin plates also has serious disadvantages, among which are the flexibility which makes it impossible to make the surfaces plane and parallel and the difficulty of measuring the thickness with the same relative degree of accuracy possible with a thicker plate.

The use of a finite plate vibrating at a harmonic frequency of one of its fundamental modes obviates many of the difficulties, for, the higher the order of the harmonic, the greater the number of standing waves between the surfaces of the plate, and consequently the smaller the wave length in comparison to the lateral dimensions of the wave. Thus, the redistribution of the stresses on a lateral surface of the plate, by crowding more waves between the large surfaces, is such that the elemental areas of stress distribution on the lateral surfaces are of much smaller size, and this means, by Saint-Venant's principle, that the boundary disturbance will be transmitted a much shorter distance into the plate. Also, the higher the order of the harmonic, the greater the difference in frequency between this harmonic and the low-frequency lateral modes of vibration, and therefore the less liklihood of coupling between modes.

The harmonic frequencies of an infinite plate are exact integral multiples of the fundamental frequency, the quantity $\left(\frac{f}{n}\right)$, equation 7, being a constant. The lower order harmonics of the finite plate, however, may differ markedly from the frequency for an infinite plate, but the higher the order of the harmonic, the closer should the value of the frequency, divided by the order of the harmonic, approach the fundamental frequency of the infinite plate of the same thickness. Thus, it should be possible to evaluate accurately the elastic constants of a precision-made finite plate by making use of accurate measurements of the frequencies of the higher order harmonics of the plate — provided, of course, that methods can be found for exciting and detecting the vibrations, and for measuring the frequency.

Specific Applications of the Theory

Piezo-electric vibrations of quartz

Crystalline quartz is a substance widely used in the communications industry because of its piezo-electric properties and mechanical stability. While the isothermal elastic constants have been measured, values given by different authors differ considerably. The adiabatic elastic constants, obtained by a dynamic method, are the constants of most value to those making use of piezo-electric quartz resonators. This investigation makes use of such a method to evaluate the elastic constants of quartz.

Since quartz exhibits piezo-electric properties, it is possible to induce vibrations in plates of the material by the application of an alternating electric field and to detect these vibrations by electrical methods. The natural facets of well-developed quartz crystals make it possible to orient the plates accurately, and the mechanical strength and hardness of the substance give it many desirable qualities.

The actual excitation of any mode of vibration by an alternating electric field depends on the energy of interaction of the electric field with the electric polarization produced within the quartz by the strains set up by the vibrations. The tensor relation between the polarization and the strains is

$$P_i = e_{ijk}\,\theta_{jk} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 13.$$

where the e's are the piezo-electric constants or moduli. Because of the symmetry of θ_{jk}, it can be seen that $e_{ijk} = e_{ikj}$.

If the alternating electric field applied to the plate is uniform, it can be shown that only the odd harmonics of the plate are excited. When standing waves are set up between the surfaces of the plate, the strains produced between loops, at any given instant, will be alternately positive and negative in the regions between the loops. Therefore, in alternate regions between nodes, the charges of polarization are of opposite sign. In the case of the fundamental (first harmonic) of a given mode of vibration, the nodal plane is equidistant from the plate surfaces, the loops being at the surfaces of the plate; and the charges produced in the vicinity of each surface are of opposite sign. There is, therefore, a reaction between the charge of polarization and the electric field. However, for the second harmonic, there are two nodal planes between the surfaces of the plate, and the region between these two nodal planes contains electric charges of sign opposite to that in the regions adjacent to the surfaces. Since the latter regions have the same charge, there is no net reaction with the electric field and the second harmonic cannot be excited electrically in this manner. Similarly, all odd harmonics of a quartz plate may be excited and all even harmonics cannot be excited by a uniform alternating electric field.

An electric field is usually applied to a quartz plate by means of two plane parallel electrodes. Sometimes electrodes of thin metal foil or metal plating are put directly on the surfaces of the plate, but these electrodes load the plate and so change its natural frequency. On the other hand, if the electrodes are arranged so that there is a finite gap between these electrodes and the quartz plate, the natural frequency of the plates may still be distorted because of the electrical properties of quartz. However, the change in frequency of a quartz plate, even for a very large electrode gap, is a small quantity, and Vigoureux[5] and Cady[6] have shown that the fractional change in frequency, caused by the electrode gap, of a harmonic of order n, is only $\frac{1}{n^2}$ times as great for the harmonic as it is for the fundamental. Therefore, the effect of the gap soon becomes negligible as the order of the harmonic is increased.

The piezo-electric moduli of quartz, defined by equation 13, are given in table I for the system of reference axes in coincidence with the natural crystallographic axes. The table also gives the relationship between two systems of constants, those constants appearing in the same rectangle being equal. Blank spaces in the table indicate that the corresponding constants are zero. According to Cady[4], $e_{11} = -e_{12} = -e_{26} = -5.10 \times 10^4$ c.g.s.e. and $e_{14} = -e_{25} = -1.35 \times 10^4$ c.g.s.e.

Table I. The polarization produced in quartz as a function of the strains

	X_x θ_{11}	y_y θ_{22}	z_z θ_{33}	y_z θ_{23},θ_{32}	z_x θ_{13},θ_{31}	x_y θ_{12},θ_{21}
P_x	e_{11}	e_{12}		e_{14}		
P_1	e_{111}	e_{122}		e_{123}, e_{132}		
P_y					e_{25}	e_{26}
P_2					e_{213}, e_{231}	e_{212}, e_{221}
P_z						
P_3						

It can be seen that modes of vibration involving the strains θ_{11}, θ_{22}, θ_{23}, and θ_{32} can be excited by alternating electric fields having a component in the x_1 direction, and that the modes involving the strains θ_{13}, θ_{31}, θ_{12}, and θ_{21} can be excited by a field component in the x_2 direction. A field in the x_3 direction excites no piezo-electric vibrations.

<u>Relationships between two systems of constants</u>

Unfortunately, the system in general use for the stress-strain and piezo-electric relationships does not conform to tensor notation. In this widely used system, the stress-strain relationship is

$$X_x = c_{11} x_x + c_{12} y_y + c_{13} z_z + c_{14} y_z + c_{15} z_x + c_{16} x_y$$
$$Y_y = c_{21} x_x + c_{22} y_y + c_{23} z_z + c_{24} y_z + c_{25} z_x + c_{26} x_y$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

where the capital letters represent stresses and the small letters total strains; y_z, for instance, being the sum of θ_{23} and θ_{32}. The elastic constants can be presented as a symmetric matrix, since $C_{21} = C_{12}$, so that for a material of no symmetry the matrix takes the form

$$
\begin{matrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
 & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\
 & & c_{33} & c_{34} & c_{35} & c_{36} \\
 & & & c_{44} & c_{45} & c_{46} \\
 & & & & c_{55} & c_{56} \\
 & & & & & c_{66}
\end{matrix}
$$

There are thus 21 independent elastic constants in a homogeneous material of no symmetry.

Quartz is a rhombohedral crystal of the trigonal holo-axial class, D_3. The x_3, or Z, axis is a three-fold axis of rotational symmetry. Three axes at right angles to the x_3 axis, any one of which can be taken as the x_1 axis, are axes of two-fold rotational symmetry. Because of the symmetry relations, quartz has only six independent elastic constants. The matrix of these constants, as given by Voigt[7,8], corresponding to the matrix for a material of no symmetry, is:

$$
\begin{matrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
 & c_{11} & c_{13} & -c_{14} & 0 & 0 \\
 & & c_{33} & 0 & 0 & 0 \\
 & & & c_{44} & 0 & 0 \\
 & & & & c_{44} & c_{14} \\
 & & & & & \frac{c_{11}-c_{12}}{2}
\end{matrix}
\qquad \text{..........} \quad 14.
$$

The relationship between the principal elastic constants and the elastic constants of the tensor notation, is presented in table II for easy reference in changing from one system to the other. As before, those constants in the same rectangle are equal to each other, and constants corresponding to blank spaces in the table are equal to zero.

<u>Orientations of plates necessary for the determination of the elastic constants</u>

In the actual work of evaluating the elastic constants of quartz, plates cut with their faces perpendicular to the natural axes of the crystal are the first to suggest themselves.

Two angles are used to designate the orientation of a quartz plate. The first angle, the azimuth, is the angle between the normal to the surface of the plate and the x_1 axis, as measured in the $x_1 x_2$ plane. The second angle, the altitude, is the angle between the normal and the $x_1 x_2$ plane.

The determinant for the $0^\circ, 0^\circ$ plate, or so-called X-cut crystal, is obtained from determinant 9 by reference to table II, from which it is seen that $C_{1121} = C_{1131} = C_{1311} = C_{1211} = 0$. Therefore, the determinant for the $0^\circ, 0^\circ$ plate is

$$\begin{vmatrix} C_{1111} - k^2 & 0 & 0 \\ 0 & C_{1221} - k^2 & C_{1231} \\ 0 & C_{1321} & C_{1331} - k^2 \end{vmatrix} = 0.$$

Since the last two subscripts of the elastic constants in the tensor notation indicate the particular strain with which the constant is associated, reference to table I shows that the

Table II

The stresses produced in quartz as a function of the strains

	x_x θ_{11}	y_y θ_{22}	z_z θ_{33}	y_z θ_{23}, θ_{32}	z_x θ_{13}, θ_{31}	x_y θ_{12}, θ_{21}
X_x θ_{11}	C_{11} C_{1111}	C_{12} C_{1122}	C_{13} C_{1133}	C_{14} C_{1123}, C_{1132}		
Y_y θ_{22}	C_{12} C_{2211}	C_{11} C_{2222}	C_{13} C_{2233}	$-C_{14}$ C_{2223}, C_{2232}		
Z_z θ_{33}	C_{13} C_{3311}	C_{13} C_{3322}	C_{33} C_{3333}			
Y_z θ_{23} θ_{32}	C_{14} C_{2311} C_{3211}	$-C_{14}$ C_{2322} C_{3222}		C_{44} C_{2323}, C_{2332} C_{3223}, C_{3232}		
Z_x θ_{13} θ_{31}					C_{44} C_{1313}, C_{1331} C_{3113}, C_{3131}	C_{14} C_{1312}, C_{1321} C_{3112}, C_{3121}
X_y θ_{12} θ_{21}					C_{14} C_{1213}, C_{1231} C_{2113}, C_{2131}	$\frac{1}{2}(C_{11}-C_{12})$ C_{1212}, C_{1221} C_{2112}, C_{2121}

strain, θ_{11}, produces a polarization in the x_1 direction, while the strains θ_{21} and θ_{31}, produce a polarization in the x_2 direction. Hence, the linear factor in the determinant represents a pure-compressional mode of vibration, excited by an alternating electric field in the x_1 direction, and the minor of this factor represents two more-complicated modes involving shear vibrations, and excited by a field in the x_2 direction.

The expansion of this determinant gives

$$[c_{1111} - k^2]\left[(c_{1221} - k^2)(c_{1331} - k^2) - c_{1321}c_{1231}\right] = 0$$

It would be expected that of the two or three modes of vibration defined by a quadratic or cubic factor in the determinant, one mode might be excited much more strongly than the other. As an example, in the last case, the piezo-electric constant, e_{221}, is larger in value than e_{231}. Of the two values of substituted into equation 8, one value would be expected to give a ratio of the A's that would make θ_{21} relatively more important than θ_{31}, and thus would produce a greater electric polarization in the quartz to be acted upon by the electric field. It is also possible that the strains may have values of opposite signs, making the net polarization very small, so that a mode of vibration would be excited very weakly, if at all.

The solution for the $30^\circ,0^\circ$ plate (or $90^\circ,0^\circ$ plate), the so-called Y-cut, can be obtained from determinant 9 by replacing the 1's of the first and fourth subscripts of the c's by

2's, since the thickness of the plate could have been taken just as well in the x_2 direction. The expansion of the resulting determinant is

$$[C_{2112}-k^2][(C_{2222}-k^2)(C_{2332}-k^2)-C_{2322}C_{2232}]=0.$$

Hence, the secular equation of the 30°,0° plate also contains a factorable term, representing a pure shear mode of vibration excited by an electric field in the x_2 direction. The other two modes of vibration can be excited by a field in the x_1 direction. (See table I.)

The solution for the 0°,90° plate (or Z-cut) can be obtained similarly by replacing the first and fourth subscripts in determinant 9 by the subscripts 3. The expansion of this determinant is

$$[C_{3113}-k^2][C_{3223}-k^2][C_{3333}-k^2]=0,$$

the determinant for this plate being completely factorable. The first factor represents a pure shear which is excited by an alternating electric field in the x_2 direction. The second factor also represents a pure shear, excited, in this case, by a field in the x_1 direction. The third factor represents a purely longitudinal mode of vibration which cannot be excited by piezo-electric methods since there is no piezo-electric constant corresponding to the θ_{33} strain. Reference to table II shows that $C_{3113}=C_{3223}$, so the natural frequencies of the first two modes of vibration are equal. However, the piezo-electric constants e_{213} and e_{123} are relatively weak, so one would not expect these two modes to be excited strongly.

The equations for plates of these three orientations, written in terms of the principal elastic constants, are, for the 0°,0° plate

$$\left[C_{11}-K^2\right]\left[\left(\frac{C_{11}-C_{12}}{2}-K^2\right)C_{44}-C_{14}^2+\left(K^2-\frac{C_{11}-C_{12}}{2}\right)K^2\right]=0, \quad\ldots\ 15.$$

for the 30°,0° plate,

$$\left[\frac{C_{11}-C_{12}}{2}-K^2\right]\left[\left(C_{11}-K^2\right)C_{44}-C_{14}^2+\left(K^2-C_{11}\right)K^2\right]=0, \quad\ldots\ 16.$$

and for the 0°,90° plate,

$$\left[C_{44}-K^2\right]\left[C_{44}-K^2\right]\left[C_{33}-K^2\right]=0 \quad\ldots\ldots\ldots\ldots\ 17.$$

Examination of equations 15, 16, and 17 shows that (with the exception of the third factor in equation 17, which represents a mode of vibration that cannot be excited) only four of the six independent elastic constants of quartz are involved in the equations. If the K^2's are determined for the first factors of equations 15 and 16, the constants C_{11} and $\frac{C_{11}-C_{12}}{2}$ are obtained immediately, and these constants can then be substituted into the second factors of these equations. Therefore, only two K^2's of the four possible for these two factors, must be determined to make it possible to solve these second factors simultaneously for C_{14} and C_{44}.

The two remaining principal elastic constants, C_{13} and C_{33}, enter the stress-strain relationship with the strain Θ_{33}. In order to obtain relationships involving both of these constants, the plate must have an orientation such that the normal to the plate is not in the x_1x_2 plane or parallel to the x_3

axis.

The solution for the 0°,45° is obtained, and this solution is found to contain the desired constants, C_{13} and C_{33}. The elastic constants are transformed to this new orientation by means of equation 10, the determinant containing these transformed constants in terms of the principal elastic constants, for the 0°,45° orientation, being

$$\begin{vmatrix} \frac{1}{4}(C_{11}+4C_{44}+2C_{13}+C_{33})-k^2 & \frac{3}{2\sqrt{2}}C_{14} & \frac{1}{4}(C_{33}-C_{11}) \\ \frac{3}{2\sqrt{2}}C_{14} & \frac{1}{2}\left(\frac{C_{11}-C_{12}}{2}+C_{44}\right)-k^2 & -\frac{1}{2\sqrt{2}}C_{14} \\ \frac{1}{4}(C_{33}-C_{11}) & -\frac{1}{2\sqrt{2}}C_{14} & \frac{1}{4}(C_{11}-2C_{13}+C_{33})-k^2 \end{vmatrix}=0$$

The λ-secular equation, equation 12, for this plate is

$$\begin{vmatrix} \frac{1}{2}(C_{11}-C_{44})-k^2 & C_{14} & \frac{1}{2}(C_{44}+C_{13}) \\ C_{14} & \frac{1}{2}\left(\frac{C_{11}-C_{12}}{2}+C_{44}\right)-k^2 & \frac{1}{2}C_{14} \\ \frac{1}{2}(C_{44}+C_{13}) & \frac{1}{2}C_{14} & \frac{1}{2}(C_{44}+C_{33})-k^2 \end{vmatrix}=0 \qquad \times ?$$

These two determinants are, of course, exactly equivalent, and one form can be transformed to the other form.

Expansion of either of these determinants gives as the equation for the 0°,45° plate

$$\left[-\frac{1}{8}\left(\frac{c_{11}-c_{12}}{2}+c_{44}\right)+\frac{1}{4}K^2\right] c_{13}^2$$

$$+\left[\frac{1}{2}c_{14}^2-\frac{1}{4}c_{44}\left(\frac{c_{11}-c_{12}}{2}+c_{44}\right)+\frac{1}{2}c_{44}K^2\right] c_{13}$$

$$+\left[\frac{1}{8}\,\frac{c_{11}-c_{12}}{2}(c_{11}+c_{44})+\frac{1}{8}c_{44}(c_{11}+c_{44})-\frac{1}{2}c_{14}^2\right.$$

$$\left.-\frac{1}{4}\left(\frac{c_{11}-c_{12}}{2}+2c_{44}+c_{11}\right)K^2+\frac{1}{2}K^4\right] c_{33}$$

$$+\frac{1}{8}\left[c_{44}\left(\frac{c_{11}-c_{12}}{2}c_{11}+c_{11}c_{44}-c_{14}^2\right)-c_{11}c_{14}^2\right]$$

$$+\left[\frac{5}{4}c_{14}^2-\frac{1}{4}\,\frac{c_{11}-c_{12}}{2}c_{11}-\frac{1}{2}c_{44}\left(c_{11}+\frac{c_{11}-c_{12}}{2}+c_{44}\right)\right]K^2$$

$$+\frac{1}{2}\left[c_{11}+\frac{c_{11}-c_{12}}{2}+3c_{44}\right]K^4-K^6=0 \quad \text{.......} \; 18.$$

This equation is unfactorable and the modes of vibration are very complex. Because of the symmetry about the x_1 crystallographic axis, this equation is also true for a 0°,-45° plate. The strains involved in this equation are such that the resulting polarization has components in both the x_1 and x_2 directions, so it would be expected that the modes of vibration could be excited by any field having a component in either or both of the directions x_1 and x_2.

If the four elastic constants c_{11}, c_{12}, c_{14}, and c_{44} are known, and if at least two of the three κ^2's can be found experimentally, equations 18 can be solved simultaneously for the two remaining constants, c_{13} and c_{33}.

Another plate inclined toward the x_3 axis, and having an orientation which can be obtained accurately, is the so-called R-cut plate. The R plate (a 30°,38°12'38", or a 90°,38°12'38" plate) is a plate whose faces are parallel to a positive rhombohedral facet, or so-called R face of the natural quartz crystal.

All quartz plates having an azimuth angle of 90°, that is, Y-cuts tilted with respect to the x_3 axis, have λ-secular equations of the form

$$\begin{vmatrix} l_2^2\frac{c_{11}-c_{12}}{2}+l_3^2c_{44}+2l_2l_3c_{14}-\kappa^2 & 0 & 0 \\ 0 & l_2^2c_{11}+l_3^2c_{44}-2l_2l_3c_{14}-\kappa^2 & -l_2^2c_{14}+l_2l_3(c_{44}+c_{13}) \\ 0 & -l_2^2c_{14}+l_2l_3(c_{44}+c_{13}) & l_2^2c_{44}+l_3^2c_{33}-\kappa^2 \end{vmatrix}=0$$

where l_2 and l_3 are the cosines of the angles between the normal to the plate and the x_2 and x_3 axes, respectively.

For the R plate, the expansion of this determinant takes the form

$$\begin{aligned}&\left[.61740\frac{c_{11}-c_{12}}{2}+.38260c_{44}+.97205c_{14}-\kappa^2\right]\cdot\\ &\left[-.23622c_{13}^2+(.60014c_{14}-.47244c_{44})c_{13}\right.\\ &\quad+(.23622c_{11}+.14638c_{44}-.37191c_{14}-.38260\kappa^2)c_{33}\\ &\quad+(.61740c_{44}-\kappa^2)(.61740c_{11}+.38260c_{44}-.97205c_{14}-\kappa^2)\\ &\quad\left.-(.48603c_{44}-.61740c_{14})^2\right]=0 \qquad \ldots\ldots\ 19.\end{aligned}$$

Here, again, the equation is factorable, the first factor representing a mode of vibration composed of a combination of the shear strains θ_{12} and θ_{13}. Therefore, this mode of vibration can be excited by a field having a component in the x_2 direction. The second factor defines two more-complicated modes of vibration involving the strains θ_{22}, θ_{33}, θ_{23}, and θ_{32}, and thus capable of being excited by a field in the x_1 direction. (table I).

EXPERIMENTAL

Preparation of the Quartz Plates

If the adiabatic elastic constants of an aeolotropic elastic substance are to be determined accurately by means of vibrating plates, it is imperative that the sample of the substance to be used be entirely free of imperfections or flaws. It is also necessary that some way be devised to orient the plates accurately and to make the faces of the plate very closely parallel.

Plates of various orientations were cut from two large, well-developed, natural quartz crystals. These two crystals in the following shall be designated as crystal 1 and crystal 2. Crystal 1 was approximately 7 centimeters in diameter and 11 cm. long in the direction of the optic axis and crystal 2 was approximately 10 cm. in diameter by 13 cm. in length.

Crystals 1 and 2 were selected on the basis of their well-developed, reflecting factes and their relative freedom from twinning. The preliminary examination of the crystals for twinning was made with the crystals immersed in nitro-benzene, a liquid whose index of refraction is very close to that of quartz. Polarized light was directed through the crystals in the direction of the optic, or x_3, axis and the visual examination made through a plate of polaroid according to the standard procedure in testing for twinning. Only portions of the crystal at a considerable distance from any twinning were used for the plates.

The quartz was cut by means of a power-driven, rotating steel disk on which a mixture of carborundum and water was fed. The quartz was ground on two power-driven, horizontal disks, one of the disks being used for coarser grinding and the other for the finer work. All surfaces were finished with hand grinding on plate glass, with various grades of fine emery as the abrasive. The final surfaces exhibited a semi-optical finish.

The R faces of the natural crystal and the z or (R') faces with which the R faces alternate, all occur with the same angle between them and the x_3 axis, terminating the crystal with a hexagonal pyramid if all faces are equally developed. However, since the R faces are almost invariably larger, they provide three convenient planes for orienting the plates.

Two accurately oriented faces, to be used as reference planes, were ground on each crystal. After the reference planes had been ground, a slice about 2.5 cm. thick was cut from the base of each crystal, and the process again repeated.

The first reference plane consisted of one of the surfaces of the slice accurately ground perpendicular to the optic (x_3) axis. To obtain this plane, the crystal was placed base down on a piece of plate glass and a telescope at one side was focused on a normal reflection in one of the R faces. Then the crystal was rotated through 120° intervals from one R face to the next, and the base of the crystal ground until all three reflections came back to the horizontal cross-hair of the telescope.

The source of light for the telescope, consisting of a small, polished steel ball, was mounted in front of the telescope lens so that the reflection of a distant incandescent lamp seemed to originate at the center of the telescope lens. This device provided almost a point source of light for normal reflection for the crystal face.

The second reference plane consisted of a surface perpendicular to the x_2, or Y, axis. An angle block was first adjusted on a plate of glass so that one surface was perpendicular to the horizontal axis of the telescope, this adjustment being made by an optical method. The second reference plane of the crystal was then ground so that the reflection of the point of light from an R face fell on the vertical cross-hair in the telescope when the reference plane was put in contact with the angle block, and at the same time no light could pass between the reference plane and the angle block. It was estimated that the two reference faces were accurate well with 1' of the specified angle of orientation.

After the slice was cut from the crystal, the side opposite the first reference plane was polished. Each slice was then examined for twinning by the use of polarized light, and all twinned areas rejected.

All of the plates were made with approximately the same dimensions. In all cases, one surface of the plate was ground and polished so that its orientation was correct to within about 1' of the specified angle before the blank for the plate was cut from the larger quartz slice. This surface for the 0°,0° plates was made perpendicular to the two reference planes, the angle being checked with a surface plate and angle block, the grinding continuing until no light could pass between the block and the quartz. The 30°,0° surface was made perpendicular to the first reference plane and parallel to the second. The 0°,45° surface was obtained by focusing the telescope on one of the 45° angles of a right angle prism, and by grinding the surface until the angle of reflection from this surface was the same as from the prism, the surface, at the same time, being perpendicular to the second reference plane. The R plates were taken parallel to the natural R facets of the crystal.

The second surface of the plate was ground parallel to the first to one ten-thousandth of a centimeter, a good micrometer caliper being used to measure the thickness. The lateral dimensions of the plate were considerably greater than their final size while the main surfaces were being ground parallel. Then as the lateral surfaces were ground to their final form, any slight rounding of the edges on the main surfaces was removed.

The lateral sides of the plate were ground accurately to conform to the natural axes of the crystal.. All angles were right angles, and opposite lateral surfaces were made parallel to each other to better than one-thousandth centimeter for those plates made from crystal 1 and to about one ten-thousandth centimeter for those plates from crystal 2. The directions of the principal axes of the crystals were marked on the lateral surfaces of the plates with indelible ink. The finished plates were free from nicks or other flaws and the sharp edges and corners were not rounded.

One useful plate of each of the orientations 0°,0°; 30°,0°; 0°,45°; and R, was made from each of the crystals 1 and 2, and a 0°,90° plate was made from crystal 1. All the plates made from a particular crystal (with the exception of the R plate from crystal 1) were cut from the same quartz slice within about 2.5 cm. of a central point. In table III is given the final dimensions of the quartz plates.

Table III. The dimensions of the quartz plates

From crystal number	Orientation of plate	Thickness in cm.	Lateral dimensions, cm.	
1	0°,0°	$x_1=.4472$	$x_2=2.202$	$x_3=2.238$
1	30°,0°	$x_2=.4413$	$x_1=2.199$	$x_3=2.138$
1	0°,90°	$x_3=.4411$	$x_1=2.243$	$x_2=2.170$
1	0°,45°	$x_1+45^\circ=.4451$	$x_2=2.229$	$x_1-45^\circ=2.202$
1	30°,38°12½'	$x_2+38^\circ=.4503$	$x_1=2.178$	$x_2-52^\circ=2.224$
2	0°,0°	$x_1=.4473$	$x_2=2.1723$	$x_3=2.2123$
2	30°,0°	$x_2=.4442$	$x_1=2.1983$	$x_3=2.1808$
2	0°,45°	$x_1+45^\circ=.4437$	$x_2=2.2238$	$x_1-45^\circ=2.2283$
2	30°,38°12½'	$x_2+38^\circ=.4364$	$x_1=2.1852$	$x_2-52^\circ=2.2170$

The density of the quartz in crystals 1 and 2 was very carefully determined and found to be the same in both cases within the experimental error. This value of the density, 2.648 grams per cubic centimeter (at 32°C), will be used in all the following work. It may be mentioned that this value of the density is in exact agreement with the value obtained by a linear interpolation between the values for the density given by Sosman[11] for 0°C and 50°C.

The Apparatus

There are several requirements for apparatus designed for the measurement of the frequencies of the higher order harmonics of quartz plates. A stable oscillator having a frequency variable over a wide range is required for the production of the alternating, radio-frequency e.m.f. necessary for exciting the vibrations in the plate. A constant-temperature chamber is necessary for keeping the temperature of the plates constant, and specially devised instruments are needed for detecting and accurately measuring the harmonic frequencies.

The oscillator consisted of a 57 pentode in an electron-coupled circuit. The frequency of the oscillator could be varied continuously between the limits of approximately 600 to 36,000 kilocycles per second. The oscillator was followed by a buffer amplifier stage using a 2A5 pentode, and this in turn was followed by the quartz plate which acted as a filter between the source of e.m.f. and the detecting instruments.

In order that the boundary condition, that there be no external stress on the surfaces of the plate, be fulfilled, the plate holder was made with two sets of vertical electrode plates so that the quartz plate could be mounted in a vertical position with its weight resting on one of its lateral surfaces. The first pair of plane parallel electrodes, spaced about .468 cm. apart, was used when the field direction was applied in the direction of the thickness of the plate. The second set of electrodes, spaced about 2.54 cm. apart, was used when the field was applied in a direction perpendicular to the thickness of the plate, the crystal plate being mounted in a vertical position and perpendicular to the electrodes. The electrodes were somewhat larger in size than the quartz plates.

The quartz plate and the plate holder were contained in a constant-temperature chamber capable of maintaining the temperature constant within about .05° C. of the value for which it was set. The constant-temperature chamber was composed of two thick-walled cast-aluminum cylindrical containers, one being placed inside the other, the two thermally insulated by means of cotton placed between them. The outer container had an accurate mercury thermostat mounted within its aluminum wall, the thermostat controlling a relay which in turn controlled the electric current in the heating coil surrounding the outer container.

The temperature was measured by means of a copper-constantan thermocouple and a potentiometer. One junction of the thermocouple was mounted directly under the quartz plate and the other junction was kept at the temperature of melting ice. The inner container was electrically insulated to prevent the loss of the high frequency current to ground due to capacitance between the electrodes and the container. It was also found necessary to insulate electrically the outer container for the highest frequency range and to ground it for other frequencies.

The inner container was made so that it could be evacuated in order to remove any effect that damping by the air might have on the vibrating surfaces of the quartz plates. However, it was found that the air damping had no detectable effect on the frequencies of the higher order harmonics of a plate. The only decrease in the amplitude of vibration due to damping by the air occurred when the plate was mounted perpendicular to the second pair of electrodes and the higher frequencies were being used, and even this decrease in amplitude was small. Therefore, for most of the work, the temperature chamber was not evacuated.

Following the crystal plate in the circuit were two detecting devices. The first consisted of a three-stage audio amplifier, modified to give best results for the particular uses to which it was put. The input to the grid of the first stage of this audio amplifier was connected to shielding placed about the lead-wires conducting the r.f. current to and from the constant temperature chamber. The sensitivity of this audio amplifier was dependent on the tuning of the circuit in the second detecting device.

The second detecting device consisted of a vacuum tube voltmeter using a 57 pentode connected as a triode and preceded by a tuned circuit to prevent the harmonics from affecting the readings. The vacuum tube voltmeter had four ranges of sensitivity, each succeeding range being about three times as sensitive as the one preceding. The input to the vacuum tube voltmeter came directly from the output electrode of the crystal plate filter. The circuit diagram of the apparatus is given in Fig. 1.

Frequencies were determined by the beat method. The source of standard frequency was a frequency meter of the oscillating type, temperature-controlled, and of a special design. This frequency meter was accurately calibrated. In this calibration, a point was determined every five kilocycles, and, since the resulting curve was only slightly curved over most of its length, a linear interpolation was carried out between the calibration points. The accuracy of the meter was such that frequencies could be determined by its use to one part in 25,000.

Method of Measuring Resonance Frequencies

If an alternating e.m.f., differing in frequency from any resonance frequency of a quartz plate, be applied to the plate, the quartz reacts with but very weak forced vibrations. The two electrodes on either side of the plate, through their capacitance, transmit the high-frequency current, and a voltage

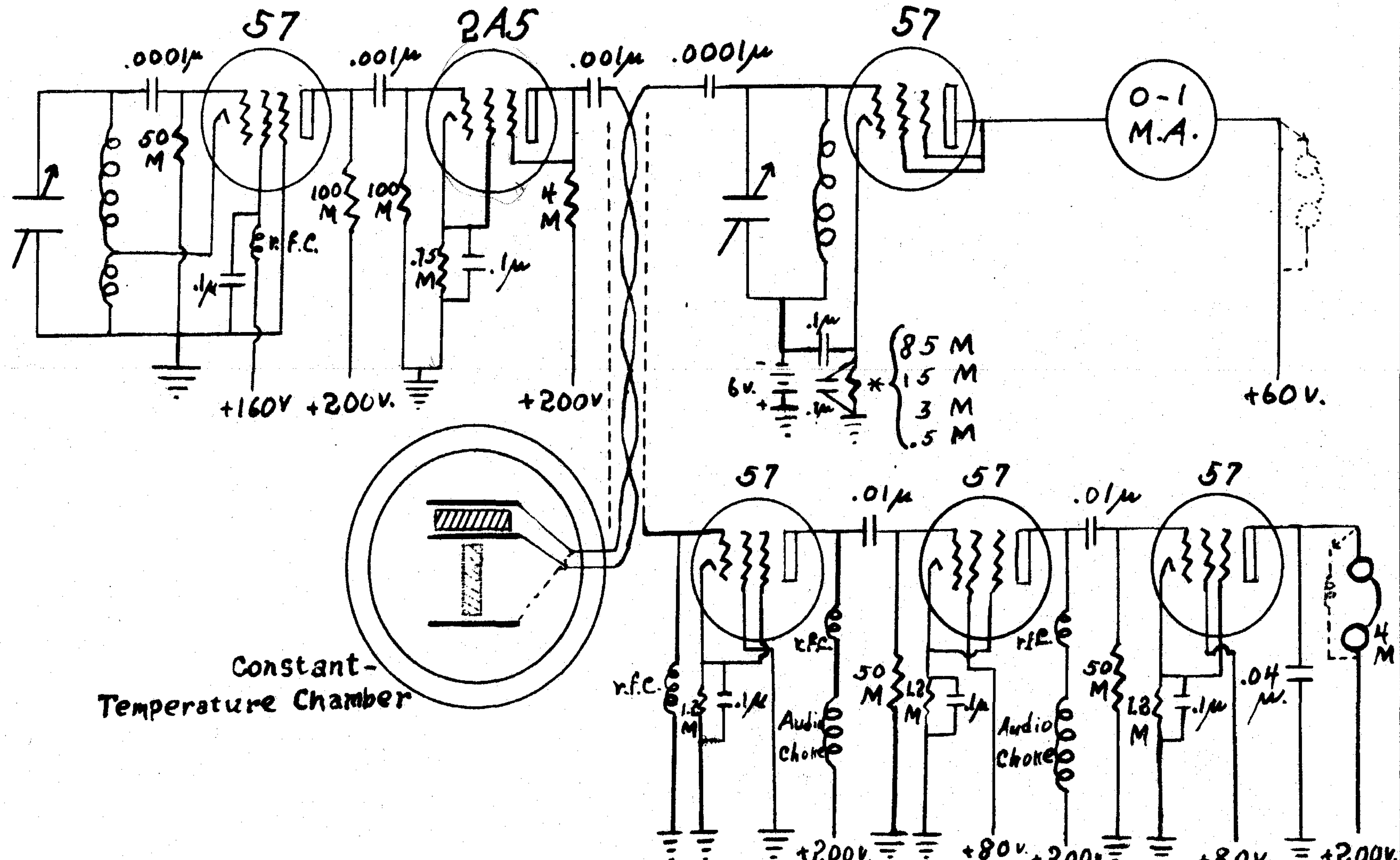

Fig. 1

Apparatus for measuring resonance frequencies measuring device detects little change because of the presence of the quartz. However, if the e.m.f. and the plate vibration are in phase, the magnitude of vibration increases greatly, the larger polarization produced in the quartz reacting with greater force with the applied field. In this case, the vibrating quartz plate absorbs energy from the circuit, and as a consequence, the vacuum tube voltmeter shows a decrease in the r.f. voltage.

In the frequency region from approximately the first to the ninth harmonics of a finite quartz plate, there are literally many hundreds of resonant frequencies for the plate, these resonance points differing widely in intensity. Beyond this frequency range the detectable frequency spectrum usually becomes continually simpler, and in general there appear only the harmonics of the main modes of vibration, or occasionally weaker "satelites" of these harmonics. Harmonics of higher order have a low intensity and the region of resonance is very sharp, so the problem of locating these harmonics is a problem of major importance.

The audio amplifying branch of the apparatus had two purposes. The first purpose was the rapid, preliminary location of resonance points. As the frequency of the oscillator in the circuit was varied with a certain degree of rapidity past a resonance frequency of the plate, (the circuit in the vacuum tube voltmeter being tuned to this frequency) a click, or a somewhat musical sound was heard in the headphones. The clicking sounds were caused by the sudden decrease and increase in the radio frequency current entering the audio amplifier. The characteristic musical sounds were produced because, as the frequency of the oscillator passed the natural frequency of the plate, the plate was set vibrating and continued to vibrate after the oscillator frequency had changed, thus producing beats between the e.m.f. induced by the direct piezo-electric effect in the plate and the oscillator frequency. As a rule, all the higher order harmonics produced clicks and only resonance points of lower frequency produced the characteristic musical sound.

The approximate location of a resonance point having been found, the oscillator was accurately tuned to the frequency of this point by means of the sharp decrease in the vacuum tube voltmeter reading at the resonance point. The audio amplifier was put into use again to obtain the beat note between the oscillator and the frequency meter in order to determine the actual frequency of the resonance point. The frequency meter, whose r.f. current was fed into the audio amplifier by a loose coupling with one of the stages of amplification, was adjusted so that as the oscillator frequency was varied slightly the same beat note was produced at points equidistant from the "bottom of the crevasse," the minimum vacuum tube voltmeter reading.

The majority of the harmonics of the quartz plates were of much higher frequency than the frequencies possible for the frequency meter, which had a range of 500 to 1050 kilocycles per second. The beat notes obtained, therefore, were between some harmonic of the frequency meter and some harmonic of the oscillator. To simplify the task of frequency determination, however, only the louder beats, those beats between the fundamental of the oscillator and some harmonic of the frequency meter, were used. An idea of the method of procedure and the type of work necessary to determine the higher frequencies can be obtained from table IV which contains data taken from actual frequency determinations of the harmonics of the 0°,0° plate from crystal 1.

Column 1 of table IV contains readings of the frequency meter when tuned to a resonance point of the quartz plate. All those readings contained in one group are for the same resonance point. Column 2 contains the frequencies, determined by the use of a linear interpolation chart, which correspond to the meter readings in column 1. In column 3 is given the order of the frequency meter harmonic which was producing beats with the fundamental of the oscillator. In column 4 is presented the product of columns 2 and 3. This number represents the actual frequency of a harmonic of the quartz plate. Since the independently determined frequencies for any given plate harmonic, as presented in this column, lie very close together, the errors in the frequency meter calibration, as well as the experimental error, must be very small. Column 5 contains the most probable frequency of column 4. The number in column 6 is the order of the harmonic of the plate, and column 7 contains the frequency

Table IV. Illustration of the method used for determining harmonic frequencies

1 Frequency meter reading	2 Corresponding frequency in KC/sec.	3 Order of frequency meter harmonic	4 Frequency of plate harmonic, in KC/sec.	5 Chosen frequency of plate harmonic, γ	6 Order of plate harmonic, n	7 $\frac{\gamma}{n}$
20.601	981.78	36	35,344			
19.272	955.25	37	35,344			
17.894	930.07	38	35,345	35,344	55	642.62
16.484	906.23	39	35,345			
15.030	883.62	40	35,345			
20.180	973.14	35	34,060			
18.785	946.10	36	34,059	34,059	53	642.62
17.343	920.52	37	34,059			
21.137	993.12	33	32,773			
19.720	963.86	34	32,771	32,773	51	642.61
16.737	910.35	36	32,773			
20.711	984.06	32	31,490			
19.216	954.15	33	31,487	31,488	49	642.61
17.668	926.07	34	31.487			
20.240	974.32	31	30,204			
18.663	943.85	32	30,203	30,203	47	642.62
17.032	915.20	33	30,202			
19.722	963.90	30	28,917			
18.056	932.87	31	28,919	28,918	45	642.62
16.326	903.68	32	28,918			
20.848	986.89	28	27,633			
19.147	952.87	29	27,633	27,633	43	642.63
17.377	921.07	30	27,632			

of vibration of the plate, divided by the order of the harmonic. The resulting number is seen to be practically a constant.

The procedure in the measurement of the harmonic frequencies of a quartz plate was first to measure the highest order harmonics that were obtained, and then to work in a consecutive manner toward the harmonics of lower frequency. In the complex, lower frequency region of the frequency "spectrum" only the frequencies of those strongest resonance points, which by their placement suggested that they might be harmonics of the modes of vibration that were desired, were measured. In the lower frequency region, there seemed to be a tendency for strong resonance points to occur in groups, and, as a rule, one of the central resonance points of the group was considerably stronger than the others. This resonance point was almost invariably a harmonic of one of the main modes of vibration of the plate.

THE RESULTS

Measurements of Harmonic Frequencies

The harmonic frequencies of all the plates were measured with the constant-temperature chamber operating at 35.2° C. After a new quartz plate had been mounted vertically in the holder in the temperature chamber, sufficient time was allowed for complete thermal equilibrium to occur.

The harmonic frequencies, ν, which were determined in this investigation are presented in tables V to XII, inclusive. These tables also record the order of the plate harmonic, n, and also the quotient of the harmonic frequency and the order of the harmonic. This quotient, $\frac{\nu}{n}$, is the number which should approximate more closely the fundamental frequency of an infinite plate of the same thickness as the order of the harmonic is increased. An experimental evidence for the truth of this prediction lies in the fact that the variation in the value of $\frac{\nu}{n}$ from one harmonic to the next is much smaller for the harmonics of higher order than for those of lower order.

The quantity $\frac{\nu}{n}$ multiplied by the thickness of the plate, d, is equal to one-half the wave velocity, and this is a constant which should be comparable between quartz plates of the same orientation if quartz is uniform from one sample to the next. Therefore, the comparative values of the wave velocities for the higher order harmonics should give a good indication of the uniformity of crystalline quartz.

As previously explained, only the harmonics of odd order were excited by the methods employed.

<u>The harmonics of the 0°,0° plates</u>

The frequencies of the measured harmonics of the 0°,0° plates from crystals 1 and 2 are presented in tables V and VI, respectively. It can be seen that the values of $\frac{\nu}{n}$ for the frequencies of lower order show more variation than do the values for the higher order harmonics. A clearer idea of the variations of $\frac{\nu}{n}$ with respect to n can be obtained from Fig. 2.

It is evident that a value of $\frac{\nu}{n}$ accurately representative of the fundamental frequency of an infinite plate of the same thickness can be chosen, since for all harmonics of higher order than the thirtieth, the variation of the values of $\frac{\nu}{n}$ from the mean value is less than one part in 10,000. It must be remembered that this variation also includes the experimental error. A comparison of the values of $d \cdot \frac{\nu}{n}$ between the plates from crystals 1 and 2 can be obtained from table XIII.

The harmonics of the mode of vibration of a 0°,0° plate, excited by an electric field in the x_1 direction is defined by the first term of equation 15. The mode excited by the x_2 field corresponds to the second bracket of the same equation. Only one of the two modes of vibration which should be excited by an

Table V. The harmonic frequencies, ν, and the values of ν/n for the 0°,0° plate from crystal 1. The frequencies are given in kilocycles per second.

Electric field in x_1 direction			Electric field in x_2 direction		
n	ν	ν/n	n	ν	ν/n
1?	638	638	1	377	377
1?	649	649	3	1,115	371.8
3	1,926	642.2	5	1,854	370.8
5	3,213	642.6	7	2,591	370.1
7	4,504	643.4	9	3,324	369.3
9	5,790	643.3	11	4,062	369.3
11	7,071	642.8	13	4,797	369.0
13	8,356	642.8	15	5,534	369.0
15	9,644	642.9	17	6,271	368.9
17	10,927	642.8	19	7,010	368.9
19	12,213	642.79	21	7,747	368.9
21	13,496	642.67	23	8,484	368.9
23	14,781	642.65	25	9,221	368.8
25	16,070	642.80	27	9,959	368.85
27	17,353	642.70	29	10,696	368.83
29	18,639	642.62	31	11,433	368.81
31	19,923	642.68	33	12,171	368.82
33	21,207	642.64	35	12,909	368.83
35	22,493	642.66	37	13,646	368.81
37	23,778	642.65	39	14,384	368.82
39	25,062	642.62			
41	26,350	642.68			
43	27,633	642.63			
45	28,918	642.62			
47	30,203	642.62			
49	31,488	642.61			
51	32,773	642.61			
53	34,059	642.62			
55	35,344	642.62			

Table VI. The harmonic frequencies, γ, and the values of γ/n for the 0°,0° plate from crystal 2. The frequencies are given in kilocycles per second.

Electric field in x_1 direction			Electric field in x_2 direction		
n	γ	γ/n	n	γ	γ/n
1?	638	638	3	1,119	373.0
1?	647	647	5	1,854	370.8
3	1,926	642.2	7	2,589	369.9
5	3,212	642.4	9	3,323	369.2
7	4,490	641.4	11	4,060	369.1
9	5,781	642.3	13	4,796	368.9
11	7,068	642.5	15	5,532	368.8
13	8,351	642.4	17	6,269	368.8
15	9,637	642.5	19	7,007	368.8
17	10,922	642.47	21	7,743	368.7
19	12,207	642.47	23	8,480	368.7
21	13,491	642.43	25	9,217	368.7
23	14,774	642.35	27	9,955	368.7
25	16,063	642.52	29	10,691	368.66
27	17,346	642.44	31	11,428	368.65
29	18,631	642.45	33	12,166	368.67
31	19,915	642.42	35	12,903	368.66
33	21,200	642.42	37	13,640	368.65
35	22,483	642.37	39	14,377	368.64
37	23,769	642.40	41	15,115	368.66
39	25,053	642.38	43	15,852	368.65
41	26,338	642.39	45	16,587	368.60
43	27,663	642.38	47	17,326	368.64
45	28,908	642.40	49	18,065	368.67
47	30,193	642.40	51	18,800	368.63
49	31,475	642.35	53	19,536	368.60
51	32,759	642.33	55	20,274	368.62
53	34,045	642.36	57	21,012	368.63
55	35,329	642.35	59	21,748	368.61
			61	22,485	368.61
			63	23,224	368.63
			65	23,959	368.60
			67	24,697	368.61
			69	25,434	368.61
			71	26,172	368.62
			73	26,908	368.60
			75	27,648	368.64
			77	28,383	368.61

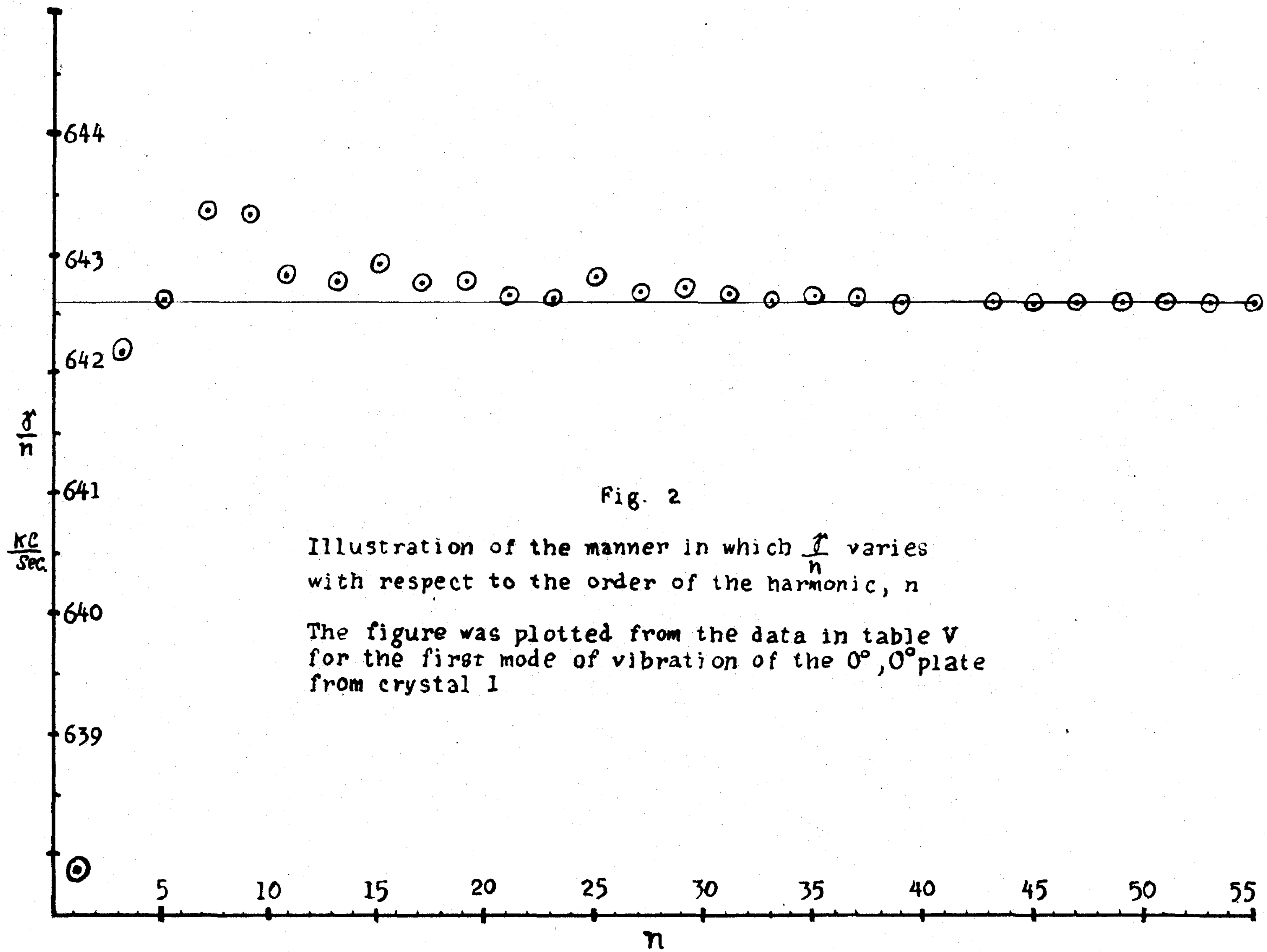

Fig. 2

Illustration of the manner in which $\frac{\gamma}{n}$ varies with respect to the order of the harmonic, n

The figure was plotted from the data in table V for the first mode of vibration of the 0°,0° plate from crystal 1

x_2 field according to theory, was obtained experimentally.

The harmonics of the 30°,0° plates

The measured harmonic frequencies of the 30°,0° plates are presented in table VII and table VIII. Here, again, the values of $\frac{\nu}{n}$ for all harmonics above the thirtieth, with a very few exceptions, lie within one one-hundredth of 1 percent of the mean value.

The mode of vibration excited by the alternating electric field in the x_2 direction, corresponds to the first term of equation 16. Both modes of vibration excited by the x_1 field were obtained experimentally, but one of these modes was excited so weakly that only a few of its harmonics could be measured, and these not with great accuracy.

The harmonics of the 0°,45° plates

Tables IX and X contain the experimental harmonic frequencies for the 0°,45° plates from crystals 1 and 2, respectively. The percentage variation of $\frac{\nu}{n}$ from the mean for all harmonics above the thirtieth is comparable to the variation in the cases already given.

All three modes of vibration for the 0°,45° plates were excited by an electric field in the direction of the thickness of the plate. These three modes were excited with approximately the same strength in the frequency spectrum so that it was impossible to distinguish the harmonics of a given mode except by

Table VII. The harmonic frequencies, ν , and the values of ν/n for the 30°,0° plate from crystal 1. The frequencies are given in kilocycles per second.

Electric field in x_2 direction			Electric field in x_1 direction		
n	ν	ν/n	n	ν	ν/n
1	448	448	1	634	634
3	1,324	441.4	3	2,047	682.4
5	2,224	444.9	5	3,402	680.4
7	3,121	445.8	7	4,763	680.4
9	4,015	446.1	9	6,121	680.1
11	4,855	441.4	11	7,481	680.1
13	5,768	443.7	13	8,842	680.1
15	6,661	444.0	15	10,201	680.1
17	7,552	444.2	17	11,560	680.1
19	8,441	444.3	19	12,921	680.05
21	9,331	444.3	21	14,281	680.05
23	10,222	444.43	23	15,639	679.96
25	11,111	444.44	25	17,000	680.00
27	12,000	444.44	27	18,357	679.89
29	12,889	444.45	29	19,719	679.97
31	13,780	444.52	31	21,078	679.94
no 33 or 35			33	22,438	679.94
37	16,442	444.38	35	23,798	679.94
39	17,333	444.44	37	25,158	679.95
41	18,222	444.44	39	26,515	679.87
43	19,111	444.44			
45	20,000	444.44	3	1,481	493.8
47	20,889	444.45	9	4,408	489.8
49	21,779	444.47	11	5,378	488.9
51	22,669	444.49	15	7,340	489.3
53	23,558	444.49	23	11,253	489.3
55	24,443	444.42	25	12,234	489.36
57	25,335	444.47			
59	26,225	444.49			
61	27,113	444.47			
63	28,003	444.49			
65	28,889	444.45			
67	29,780	444.48			
69	30,670	444.49			
71	31,559	444.49			
73	32,447	444.48			
75	33,336	444.48			
77	34,222	444.44			
79	35,113	444.47			

Table VIII. The harmonic frequencies, ν, and the values of ν/n for the 30°,0° plate from crystal 2. The frequencies are given in kilocycles per second.

Electric field in x_2 direction			Electric field in x_1 direction		
n	ν	ν/n	n	ν	ν/n
1	452	452	1	679	679
3	1,314	438.0	3	2,034	678.2
5	2,214	442.9	5	3,380	676.1
7	3,100	442.9	7	4,731	675.9
9	3,988	443.1	9	6,083	675.8
11	4,877	443.4	11	7,433	675.7
13	5,730	440.8	13	8,783	675.6
15	6,615	441.0	15	10,135	675.67
17	7,499	441.2	17	11,485	675.59
19	8,384	441.2	19	12,837	675.63
21	9,266	441.3	21	14,189	675.67
23	10,151	441.35	23	15.538	675.57
25	11,034	441.36	25	16,892	675.68
27	11,917	441.37	27	18,242	675.63
29	12,798	441.31	29	19,592	675.59
31	13,683	441.39	31	20,944	675.61
33	14,559	441.18	33	22,294	675.58
35	15,446	441.31	35	23,645	675.57
37	16,330	441.35	37	24,997	675.59
39	17,214	441.38	39	26,347	675.56
41	18,097	441.39	41	27,699	675.59
43	18,980	441.39	43	29,050	675.58
45	19,863	441.40	45	30,402	675.60
47	20,745	441.38			
49	21,625	441.33	3	1,470	490
51	22,507	441.31	9	4,377	486.4
53	23,390	441.32			
55	24,274	441.35			
57	25,155	441.32			
59	26,039	441.34			
61	26,920	441.31			
63	27,804	441.33			
65	28,687	441.34			
67	29,569	441.33			
69	30,450	441.30			
71	31,334	441.32			
73	32,219	441.36			
75	33,099	441.32			
77	33,990	441.43			
79	34,866	441.34			
81	35,748	441.33			

Table IX. The harmonic frequencies, ν, and the values of ν/n for the $0^\circ, 45^\circ$ plate from crystal 1. Frequencies are given in kilocycles per second. The electric field was in the direction of the thickness of the plate.

First mode of vibration			Second mode of vibration		
n	ν	ν/n	n	ν	ν/n
1	747	747			
3	2,248	749.3	5	2,445	489.0
5	3,746	748.1	7	3,429	489.9
7	5,243	749.1	9	4,405	489.5
9	6,742	749.1	11	5,384	489.5
11	8,239	749.0	13	6,362	489.4
			15	7,340	489.3
17	12,733	749.00	17	8,320	489.4
			19	9,297	489.3
21	15,729	749.00	21	10,276	489.33
23	17,226	748.96	23	11,255	489.35
25	18,725	749.00	25	12,232	489.28
27	20,221	748.93	27	13,212	489.31
29	21,720	748.97	29	14,190	489.31
31	23,218	748.97	31	15,167	489.26
33	24,715	748.94	33	16,145	489.24
35	26,214	748.97	35	17,124	489.26
37	27,711	748.95	37	18,102	489.26
39	29,210	749.03	39	19,082	489.27
41	30,707	748.95	41	20,059	489.24
43	32,203	748.91	43	21,039	489.28
45	33,701	748.91	45	22,015	489.22
47	35,198	748.89	47	22,994	489.23
			51	24,953	489.27
			53	25,931	489.26
			55	26,910	489.27
			57	27,887	489.25
			59	28,867	489.27
			61	29,845	489.26
			63	30,823	489.25
			65	31,799	489.22
			67	32,782	489.27
			69	33,757	489.23
			71	34,735	489.23
			73	35,714	489.23

Table IX. (Cont'd)

Third mode of vibration		
n	γ	γ/n
1	410	410
3	1,241	413.8
7	2,878	411.1
9	3,697	410.8
11	4,522	411.1
13	5,341	410.9
15	6,163	410.9
17	6,985	410.9
19	7,804	410.8
21	8,626	410.8
23	9,448	410.8
25	10,269	410.76
27	11,092	410.81
29	11,912	410.76
31	12,733	410.74
33	13,554	410.73
35	14,376	410.73
37	15,197	410.72
39	16,020	410.77
41	16,839	410.71
43	17,662	410.74
45	18,482	410.71
47	19,302	410.68
49	20,124	410.69
51	20,945	410.69
53	21,769	410.74
55	22,589	410.71
57	23,410	410.70
59	24,232	410.71
61	25,052	410.69
63	25,875	410.71
65	26,694	410.68
67	27,515	410.67
69	28,336	410.67
71	29,159	410.69
73	29,978	410.66
79	32,443	410.67
81	33,264	410.67
83	34,087	410.69
85	34,909	410.69
87	35,730	410.69

Table X. The harmonic frequencies, ν , and the values of ν/n for the 0°,45° plate from crystal 2. Frequencies are given in kilocycles per second. The electric field was in the direction of the thickness of the plate.

First mode of vibration			Second mode of vibration		
n	ν	ν/n	n	ν	ν/n
1	747	747	1	484	484
3	2,256	752.0	3	1,468	489.3
5	3,756	751.2	7	3,443	491.7
7	5,257	751.0	9	4,424	491.6
9	6,759	751.0	11	5,408	491.6
11	8,260	750.9	13	6,390	491.5
13	9,762	750.9	15	7,372	491.4
15	11,264	750.93	17	8,357	491.6
17	12,765	750.88	19	9,338	491.5
19	14,267	750.89	21	10,320	491.43
21	15,770	750.95	23	11,304	491.48
23	17,270	750.87	25	12,286	491.44
25	18,775	751.00	27	13,269	491.44
27	20,273	750.85	29	14,252	491.45
29	21,776	750.90	31	15,238	491.45
31	23,227	750.87	33	16,217	491.42
33	24,779	750.88	35	17,201	491.46
35	26,280	750.86	37	18,183	491.43
37	27,783	750.89	39	19,166	491.44
39	29,283	750.85	41	20,150	491.46
41	30,784	750.83	43	21,132	491.44
43	32,287	750.86	45	22,114	491.42
45	33,788	750.84	47	23,098	491.45
47	35,291	750.87	49	24,080	491.43
			51	25,063	491.43
			53	26,045	491.42
			55	27,027	491.40
			57	28,010	491.40
			59	28,992	491.39
			61	29,975	491.38
			63	30,957	491.38
			65	31,941	491.40
			67	32,921	491.36
			69	33,906	491.39
			71	34,890	491.41

Table X. (Cont'd)

	Third mode of vibration	
n	ν	ν/n
1	404	404
3	1,243	414.3
5	2,068	413.6
7	2,886	412.3
9	3,712	412.4
11	4,537	412.4
13	5,359	412.2
15	6,183	412.2
17	7,372	412.2
19	7,830	412.1
21	8,655	412.14
23	9,479	412.13
25	10,303	412.12
27	11,128	412.15
29	11,953	412.17
31	12,776	412.13
33	13,501	412.15
35	14,423	412.09
37	15,248	412.11
39	16,073	412.13
41	16,898	412.16
43	17,721	412.12
45	18,544	412.09
47	19,368	412.09
49	20,194	412.12
51	21,017	412.10
53	21,841	412.09
55	22,664	412.07
57	23,490	412.11
59	24,315	412.08
61	25,138	412.10
63	25,961	412.08
65	26,765	412.08
67	27,610	412.09
69	28,440	412.17
71	29,260	412.11
73	30,088	412.16
75	30,913	412.17
77	31,730	412.08
79	32,553	412.06
81	33,377	412.06
83	34,204	412.09
85	35,028	412.09

means of the calculated frequencies. Since only two modes of vibration, defining two K^2's, were necessary in order to solve equations 18 for c_{13} and c_{33}, the third mode of vibration provided a convenient check on the accuracy of the results.

An electric field perpendicular to the normal to the $0°,45°$ plates, in the direction of x_2, also excited all three modes of vibration. No difference in the frequencies of the harmonics when excited by the electric field in the x_2 direction and when in the direction of the thickness of the plate could be detected, except with the harmonics of very lowest order. This is thus an indication that the use of widely spaced electrodes makes no difference in the value of $\frac{\nu}{n}$ which is approached by the harmonics of higher order.

The harmonics of the R plates

The frequencies of the measured harmonics of the R plates are presented in tables XI and XII. In this case also the values of $\frac{\nu}{n}$ for the harmonics above the thirtieth almost invariably lie within one one-hundredth of one percent of the mean frequency.

The mode of vibration which was excited with the field in the direction of the thickness of the plate corresponds to the first term of equation 19, and the mode excited by the x_1 field, to the second term. Only one of the two modes defined by the second term of the equation was obtained experimentally.

In the case of the R plate from crystal 2, only a very

Table XI. The harmonic frequencies, ν, and the values of ν/n for the R plate from crystal 1. Frequencies are given in kilocycles per second.

Electric field in direction of thickness			Electric field in x_1 direction		
n	ν	ν/n	n	ν	ν/n
1	569	569	1?	491	491
3	1,651	550.3	3	1,344	448.1
5	2,758	551.5	5	2,232	446.5
7	3,825	546.4	7	3,125	446.4
9	4,939	549.3	9	4,014	446.0
11	6,042	549.3	11	4,903	445.8
13	7,143	549.4	13	5,794	445.7
15	8,243	549.5	15	6,695	445.6
17	9,345	549.7	17	7,575	445.6
19	10,432	549.1	19	8,466	445.6
21	11,540	549.5	21	9,357	445.6
23	12,639	549.5	23	10,247	445.54
25	13,740	549.60	25	11,139	445.56
27	14,840	549.63	27	12,030	445.56
29	15,937	549.55	29	12,920	445.52
31	17,038	549.61	31	13,811	445.52
33	18,136	549.58	33	14,701	445.51
35	19,236	549.59	35	15,591	445.46
37	20,332	549.51	37	16,484	445.51
39	21,437	549.67	39	17,373	445.46
41	22,531	549.54	41	18,263	445.44
43	23,632	549.58	43	19,155	445.47
45	24,728	549.51	45	20,046	445.47
47	25,829	549.55	47	20,938	445.49
49	26,927	549.53	49	21,825	445.41
51	28,025	549.51	51	22,719	445.47
53	29,127	549.57	53	23,609	445.45
55	30,224	549.53	55	24,501	445.47
57	31,322	549.51			
59	32,423	549.54			

Table XII. The harmonic frequencies, ν , and the values of ν/n for the R plate from crystal 2. Frequencies are given in kilocycles per second.

Electric field in direction of thickness			Electric field in x_1 direction		
n	ν	ν/n	n	ν	ν/n
1	561	561	1	467	467
3	1,715	571.5	3	1,397	465.7
5	2,839	567.8	5	2,301	460.2
7	3,947	563.8	7	3,221	460.07
9	5,096	566.2	9	4,141	461.11
11	6,232	566.6	11	5,058	459.82
13	7,368	566.8	13	5,980	460.00
15	8,504	566.9			
17	9,639	567.0			
19	10,770	566.8			
21	11,905	566.9			
23	13,040	566.96			
25	14,174	566.96			
27	15,308	566.96			
29	16,442	566.97			
31	17,577	567.00			
33	18,714	567.09			
35	19,848	567.09			
37	20,984	567.14			
39	22,116	567.08			
41	23,250	567.07			
43	24,385	567.09			
45	25,520	567.11			
47	26,651	567.04			
49	27,786	567.06			
51	28,921	567.08			
53	30,056	567.09			
55	31,191	567.11			
57	32,324	567.09			
59	33,457	567.07			
61	34,591	567.07			
63	35,726	567.08			

few of the harmonics of the mode of vibration excited by the x_2 field were of sufficient intensity to be measurable. The reason for this was not apparent. One surface of the quartz plate contained a minute flaw, originating from the natural crystal face, but since this flaw seemingly had no effect on the first mode of vibration, it seemed improbable that the flaw would affect the second mode. Reference to the value of $d \cdot \frac{f}{n}$, table XIII, also shows that the frequency of the supressed mode of crystal 2 was not changed appreciably.

<u>The harmonics of the 0°,90° plate</u>

According to equation 17, it should be possible to excite two modes of vibration for the 0°,90° plate, one by an electric field in the x_2 direction and the other by a field in the x_1 direction. However, these two modes could not be obtained experimentally for the plate which was cut from crystal 1. A few very weak resonance points were located in the frequency region below 10,000 kilocycles per second, but the frequencies of these few resonance points had no simple relationship between them. It may be that the modes of vibration were not excited strongly enough to be detected, or it may be that the plate had an invisible flaw which prevented it from vibrating. Since only one plate of this orientation was made, no check on its behavior was obtained.

Lateral Modes of Vibration

The quartz plates used in this investigation were rectangular parallelepipeds. Since opposite lateral surfaces of the plates were plane and parallel to each other, it would be expected that for electric fields in certain directions, standing waves might be set up between these lateral faces. It was found experimentally that certain lateral modes were excited, and the harmonic frequencies of several of these modes were measured.

Standing waves between the lateral faces perpendicular to the x_2 axis of a $0°,0°$ quartz plate are defined by equation 16, since these lateral surfaces effectively define a $30°,0°$ "plate." If the electric field is applied in the x_1 direction, one would expect one or two of the modes of vibration, defined by the second factor of equation 16, to be excited. One of these modes was found to be excited strongly enough to be measurable. Figure 3 contains the value of $\frac{\nu}{n}$ plotted against the order of the harmonic for this lateral mode of the $0°,0°$ plate from crystal 1. Because the main mode of vibration of the plate, defined by the first factor of equation 15, is much more strongly excited by an x_1 field than the lateral mode, only those harmonics of the lateral mode in frequency regions free from strong resonance points could measured.

Similarly, a lateral mode of vibration between the x_1 faces

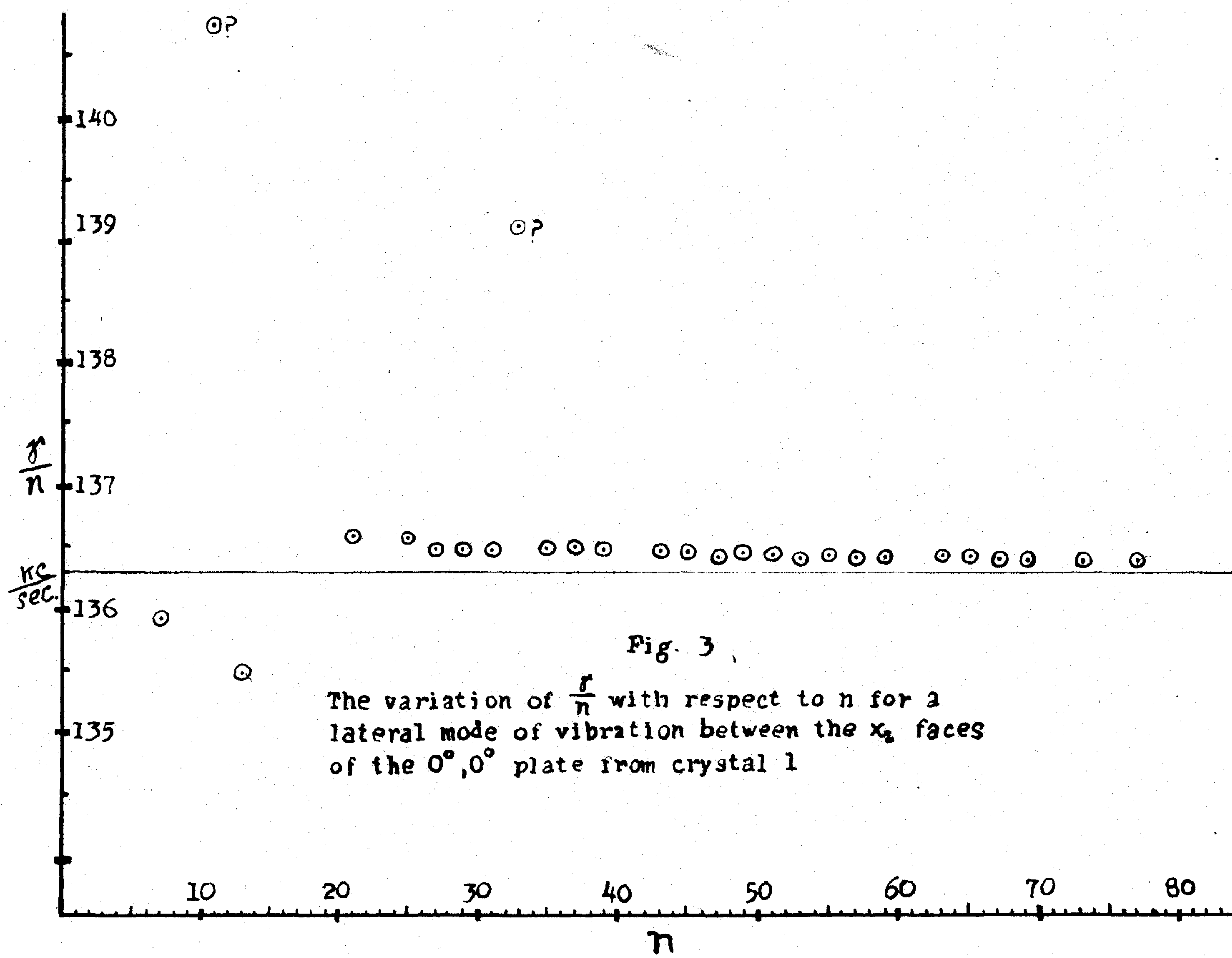

Fig. 3

The variation of $\frac{\gamma}{n}$ with respect to n for a lateral mode of vibration between the x_2 faces of the $0°,0°$ plate from crystal 1 of the 30°,0° and R plates, excited by a field in the x_2 direction, is defined by the second factor of equation 15. Harmonics of this mode of vibration, corresponding to the second mode of the 0°,0° plate, were measured for both the 30°,0° and R plates from crystal 1 and the two were found to be very similar. Figure 4 contains a plot of $\frac{\mathcal{F}}{n}$ as a function of n for this lateral mode of the 30°,0° plate.

Figures 3 and 4, which were plotted for lateral modes of vibration, in comparison to Figure 2, plotted for the main mode, clearly show that the harmonic frequencies between narrow lateral surfaces are much more perturbed than are the harmonic frequencies between broader surfaces. This would be expected because the material between the narrow surfaces is a very poor approximation to an infinite plate. Table XIII gives the values of $d \cdot \frac{\mathcal{F}}{n}$ for the lateral modes of vibration, using the values of $\frac{\mathcal{F}}{n}$ for the highest frequencies obtained, in comparison with the corresponding modes of vibration between broader surfaces. The differences existing between $d \cdot \frac{\mathcal{F}}{n}$ for lateral modes and the corresponding main modes is due to the fact that the value of $\frac{\mathcal{F}}{n}$ for the lateral modes had not yet reached the asymptotic value toward which these modes were tending.

According to theory, the coupling between modes of vibration whose unperturbed frequencies lie close together, is one of the major reasons why $\frac{\mathcal{F}}{n}$ for a finite plate differs from the value for an infinite plate. If the frequencies of all the

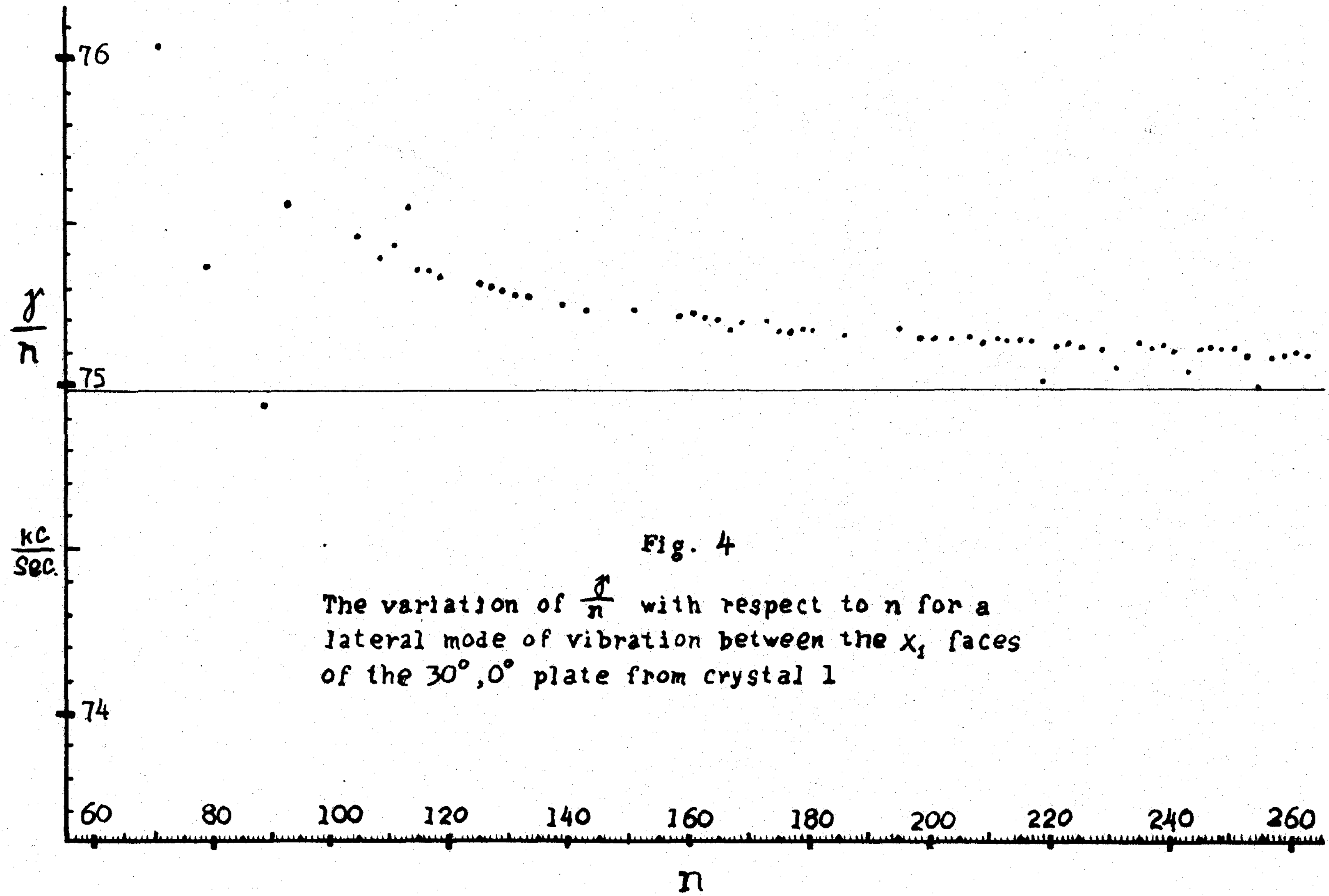

Fig. 4

The variation of $\frac{\gamma}{n}$ with respect to n for a lateral mode of vibration between the X_1 faces of the 30°,0° plate from crystal 1 natural modes of vibration and the harmonics of these modes were known, it should be possible to see which resonance points have perturbing effects on each other. Since only a few of the many natural modes of vibration of the plates were measured experimentally, a complete picture of the perturbing action of one harmonic upon another cannot be obtained from the data here presented; but in numerous cases the source of the perturbation is clear. In Figure 5, the values of $\frac{f}{n}$ are plotted as a function of frequency for the two main modes and the one lateral mode of vibration whose frequencies were experimentally measured. In the figure, the origins of the coordinate systems for the three modes has been displaced along the ordinate so that the three asymptotic values of $\frac{f}{n}$ lie close together. Vertical lines connect those resonance points which differ in frequency by only a small amount.

Calculation of the Elastic Constants

In order to evaluate the K^2's for substitution into equations 15, 16, 18, and 19 for the calculation of the elastic constants, the values of $\frac{f}{n}$ approached by the harmonics of higher order, were chosen for all modes of vibration. Table XIII gives a summary of the experimentally determined values of $\frac{f}{n}$ and also the calculated constants, $d \cdot \frac{f}{n}$ and K^2. The results for corresponding modes of vibration are grouped together for the purpose of comparison.

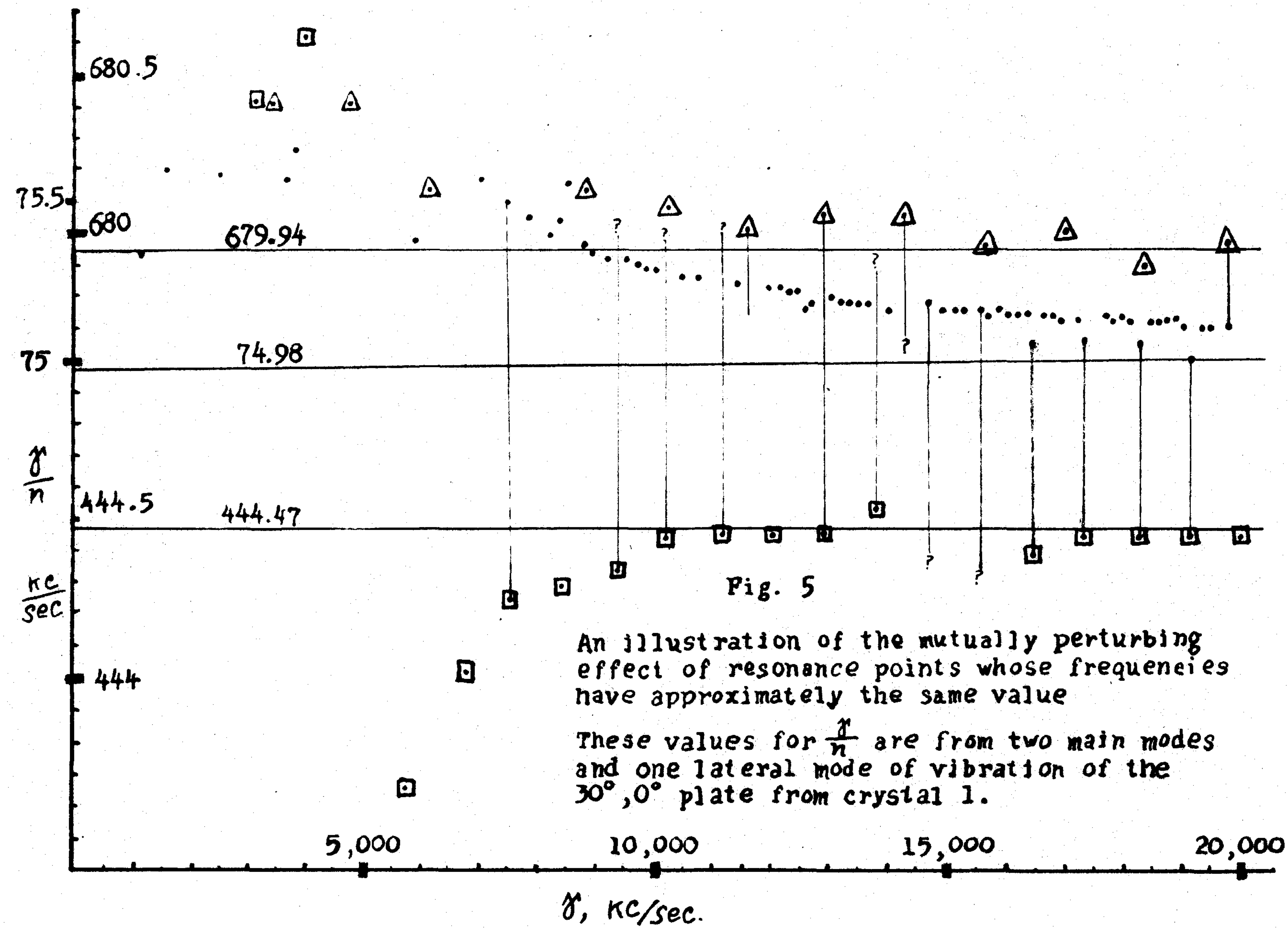

Fig. 5

An illustration of the mutually perturbing effect of resonance points whose frequeneies have approximately the same value

These values for $\frac{\gamma}{n}$ are from two main modes and one lateral mode of vibration of the 30°,0° plate from crystal 1.

Table XIII. Summary of results, including a calculation of the constants $d \cdot \frac{f}{n}$ and k^2

Crystal number	Orientation of plate	Mode of vibration	Field direction	Most probable f/n, in KC/sec.	Plate thickness, cm.	$d \cdot f/n$ in cm.KC/sec.	$k^2 = 4\rho d^2 \left(\frac{f}{n}\right)^2$ dynes/sq.cm.
1	0°,0°	x_1	x_1	642.63	.4472	287.38	87.48 x 10^{10}
2	0°,0°	x_1	x_1	642.38	.4473	287.34	87.45 "
1	0°,0°	x_1	x_2	368.82	.4472	164.94	28.82 "
2	0°,0°	x_1	x_2	368.61	.4473	164.88	28.80 "
**1	30°,0°	x_1	x_2	75.11	2.199	165.2	
**1	R	x_1	R	75.81	2.178	165.1	
1	30°,0°	x_2	x_2	444.47	.4413	196.14	40.75 "
2	30°,0°	x_2	x_2	441.33	.4442	196.04	40.71 "
1	30°,0°	x_2	x_1	679.94	.4413	300.05	95.36 "
2	30°,0°	x_2	x_1	675.58	.4442	300.09	95.38 "
**1	0°,0°	x_2	x_1	136.42	2.202	300.4	
1	30°,0°	x_2	x_1	489.4?	.4413	216.0	49.42? "
2	30°,0°	x_2	x_1	486.4?	.4442	216.1	49.44? "
1	0°,45°	Thickness		748.94	.4451	333.35	117.70 "
2	0°,45°	"	"	750.86	.4437	333.16	117.56 "
1	0°,45°	"	"	489.25	.4451	217.77	50.23 "
2	0°,45°	"	"	491.40	.4437	218.03	50.35 "
1	0°,45°	"	"	410.69	.4451	182.80	35.39 "
2	0°,45°	"	"	412.08	.4437	182.84	35.41 "
1	R	R	R	549.56	.4503	247.47	64.87 "
2	R	R	R	567.08	.4364	247.47	64.87 "
1	R	R	x_1	445.47	.4503	200.64	42.64 "
2	R	R	x_1	460. ?	.4364	200.7 ?	42.68? "

**Lateral modes of vibration

Column 1 of table XIII gives the number of the natural quartz crystal from which the particular plate was cut. Column 2 gives the orientation of the plate. Column 3 gives the direction normal to the faces between which the standing waves occur, and of course this direction is also the normal to the large faces of the plate in all cases except for the lateral modes of vibration. Column 4 gives the direction of the electric field used to excite the mode of vibration. Column 5 gives the values chosen for $\frac{f}{n}$, column 6 the thickness of the plate and column 7, $d\cdot\frac{f}{n}$, i.e. the product of columns 5 and 6, this number being one-half the wave velocity.

Since it was possible to determine $\frac{f}{n}$ to a greater degree of accuracy than was possible in the measurement of the thickness of the plate, the limiting factor in the accuracy of the calculated wave velocity, and therefore in the constant, k^2, column 8, is the measured thickness of the quartz plate. Most of the values of $d\cdot\frac{f}{n}$ for corresponding modes of vibration, when compared between plates from crystals 1 and 2 are the same within the experimental error of the thickness measurements, and therefore the conclusion is that the two quartz crystals were homogeneous, the elastic properties being the same within experimental error. The wave velocities for the $0^\circ, 45^\circ$ plates do not check quite as closely because the orientation of the plate from crystal 1 was probably not obtained quite as accurately as the plate from crystal 2. However, since the results from the two crystals in most cases are so nearly the same, a determination of the elastic constants by means of a set of plates from one crystal (crystal 2) is all that is necessary.

The first factor of equation 15 gives for one of the principal elastic constants, $C_{11} = 87.45 \times 10^{10}$ dynes per square centimeter. The first factor of equation 16 gives $\frac{C_{11} - C_{12}}{2} = 40.71 \times 10^{10}$ and therefore $C_{12} = 6.03 \times 10^{10}$. A substitution of these constants and the appropriate values of K^2 into the second factors of equations 15 and 16 and the first factor of equation 19, provides three equations in only two unknowns, C_{14} and C_{44}. The simultaneous solution of different pairs of these equations for C_{14} and C_{44} gives slightly different answers. However, the three equations 18 and the second term of equation 19 provide four more independent equations. Because of the complexity of these four equations, it is not practical to solve them simultaneously for the four constants C_{14}, C_{44}, C_{13}, and C_{33}, but these equations can be solved better by a process of successive approximations. Values for C_{14} and C_{44} were adjusted until the solution of all pairs of the four equations gave the same values for C_{13} and C_{33}. It was found that the values of C_{14} and C_{44}, as determined by this process of successive approximations using equations 18 and the second term of 19, lay between the three sets of values determined by the second terms of equations 15 and 16 and the first term of equation 19.

The values of the principal elastic constants of quartz, determined dynamically by the use of infinite plate theory and the data here presented, are given in table XIV.

Table XIV. The principal adiabatic elastic constants of quartz at 35.2° C., determined dynamically by the use of infinite plate theory

Also a comparison of quoted values of the isothermal and adiabatic constants from other sources

	(adiabatic)	Voigt[8] (isothermal)	Scheibe[10] (adiabatic)	Sosman[11] (isothermal)
c_{11}	87.55	86.8	85.46	85.1
c_{12}	6.07	7.1	7.25	6.9
c_{14}	17.25	17.2	16.82	16.8
c_{44}	57.19	58.2	57.12	57.1
c_{13}	13.2	14.4	14.35	14.0
c_{33}	106.8	107.5	105.62	105.3

CONCLUSIONS

1. The behavior of a vibrating infinite plate can be closely approximated by the use of the harmonics of higher order of a finite plate. The formulas developed with infinite plate theory hold accurately for the higher order harmonics of a finite plate.
2. The perturbation of the natural frequencies of vibration of a finite plate is caused largely by coupling with lateral modes of vibration. The perturbation is much larger for the harmonics of lower order than for the harmonics of higher order.
3. Modes of vibration produced between small surfaces distantly spaced poorly approximate the same modes for an infinite plate.
4. Different specimens of flaw-free quartz crystals have very similar elastic properties.
5. The adiabatic elastic constants of a homogeneous piezo-electric substance can be determined accurately through measurement of the frequencies of higher order harmonics of finite plates and through the application of infinite plate theory.
6. If methods could be devised for inducing, detecting, and measuring the natural vibrational frequencies of the har-

monics of plates of non-piezo-electric aeolotropic materials, the adiabatic elastic constants of these materials could also be determined by the use of infinite plate theory.

SUMMARY

The relationships existing between the elastic constants and the frequencies of the natural modes of vibration of a finite plate of homogeneous material have never received a rigorous formulation. On the other hand, the solution of the relationships between elastic constants and natural frequencies of an infinite plate is rigorous.

In the evaluation of the elastic constants of a material, an infinite plate can be approximated, either by making use of a large, thin quartz plate or by using harmonics of the fundamental modes of vibration. The latter method was used in this investigation because of its greater accuracy.

When a crystal plate is vibrating with a fundamental mode or a harmonic of this mode, the wave planes of the standing waves which are produced are parallel to the surfaces of the plate. Thus, as the order of the harmonic becomes progressively higher, more wave planes are crowded between the surfaces of the plate, and so the size of the plate becomes larger in comparison with each wave length, approximating ever closer the theoretical infinite plate.

The six independent adiabatic elastic constants of quartz were evaluated by frequency measurements of piezo-electrically excited harmonic vibrations of finite quartz plates placed in

a filter circuit in a specially designed apparatus. The approximation of the behavior of the finite plates to the behavior of infinite plates of the same thickness was excellent for harmonics of higher order than the thirtieth for the particular plates used. The frequencies of harmonics up to the eighty-seventh were measured for the main modes of vibration, and up to the two hundred sixty-third harmonic for certain lateral modes.

The quartz plates were cut from two well-developed, flaw-free quartz crystals. An optical method which made use of light reflected from the facets of the natural crystals, was used to get the orientation of the plates correct within 0° 1' of the specified angle. The two surfaces of the plates were ground plane and parallel to each other within .0001 cm.

All angles of the plates were accurate right angles and opposite sides were parallel to four significant figures. Four useful plates were cut from each of the natural crystals, these plates having the following orientations; 0°,0°; 30°,0°; 0°,45°; and 30°,38°12½'. The mathematical relationships between the elastic constants and the natural frequencies of vibration for infinite plates of each of these orientations were derived.

The adiabatic elastic constants of quartz as evaluated by this method differ significantly from the accepted elastic constants of quartz.

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